USE OF FOURIER SERIES FOR THE ANALYSIS OF BIOLOGICAL SYSTEMS

E. O. ATTINGER, A. ANNÉ, *and* D. A. MC DONALD

From the Research Institute, Presbyterian-University of Pennsylvania Medical Center, the Electromedical Division of the Moore School, and the Department of Physiology of the School of Veterinary Medicine, University of Pennsylvania, Philadelphia

ABSTRACT In an attempt to quantitate the physical behavior of biological systems, Fourier analysis has been applied to the respiratory and circulatory systems by a number of investigators. The validity of this application has been questioned on the basis that these systems are nonlinear and not strictly periodic. If these objections were valid much of the more recent work in this field would have to be re-evaluated. The applicability of Fourier analysis to these two systems was therefore investigated, both theoretically and experimentally, using on-line analysis on a LINC (laboratory instrument computer) digital computer. In normal anesthetized dogs errors introduced by deviations from periodicity and linearity were found to be within the range of measurement errors. In sinusoidally perfused aortas the amount of second harmonic produced by the vessel was less than 5%. In addition, the magnitude of errors due to faulty determination of cycle length, sampling techniques, aliasing, and A-D (analogue to digital) conversion were evaluated and found to be within the noise level of the measuring equipment when appropriate techniques were employed. Utmost care has to be used in the coupling between a transducer and the system to be measured, and dynamic calibration before each experiment is a prerequisite for successful analysis. With presently available equipment the static measurement errors can be reduced to ± 0.2 cm H_2O for pressure transducers, 0.1 cm^3/sec for electromagnetic flowmeters, and 5×10^{-4} cm for measurement of radius changes. The frequency response of this equipment once properly coupled to the system is flat to at least 20 cycle/sec.

The introduction of analytical methods and the availability of better measuring equipment in biological research have added much to a better understanding of biosystems and their behavior. The analysis of a biosystem aims at its quantitative description, which permits the prediction of its behavior under a variety of circumstances. The relations between pressure and flow or between stress and strain are examples of functions which characterize the dynamic behavior of the cardiovascular and respiratory systems. However, such relations vary in general with time and space and are considerably more complex than their counterparts in man-made systems.

The first problem in any analytical procedure is to express the measured variable as a number. Classically this has been done by the use of mean values, i.e. integration with respect to time or space, for example: cardiac output, diffusion capacity, etc. However, particularly with regard to frequency-dependent phenomena, such an approach may obscure important aspects in the function of a system.

There are a number of mathematical techniques which make it possible to express any arbitrary function of time as an infinite series. For periodic and quasi-periodic phenomena, such as encountered in the cardiovascular and respiratory systems, the so-called Fourier series are particularly suited. Although they have been introduced into circulatory physiology some 40 years ago by Frank (7), their validity has been repeatedly questioned, primarily for two reasons: cardiac action is not strictly periodic and the cardiovascular system is not linear. If these objections are valid, much of the recent work on vascular and respiratory mechanics would have to be re-evaluated. Since we art not aware of an experimental study of this problem, we decided to examine the applicability of Fourier analysis in the respiratory and circulatory systems both theoretically and experimentally.

THEORY

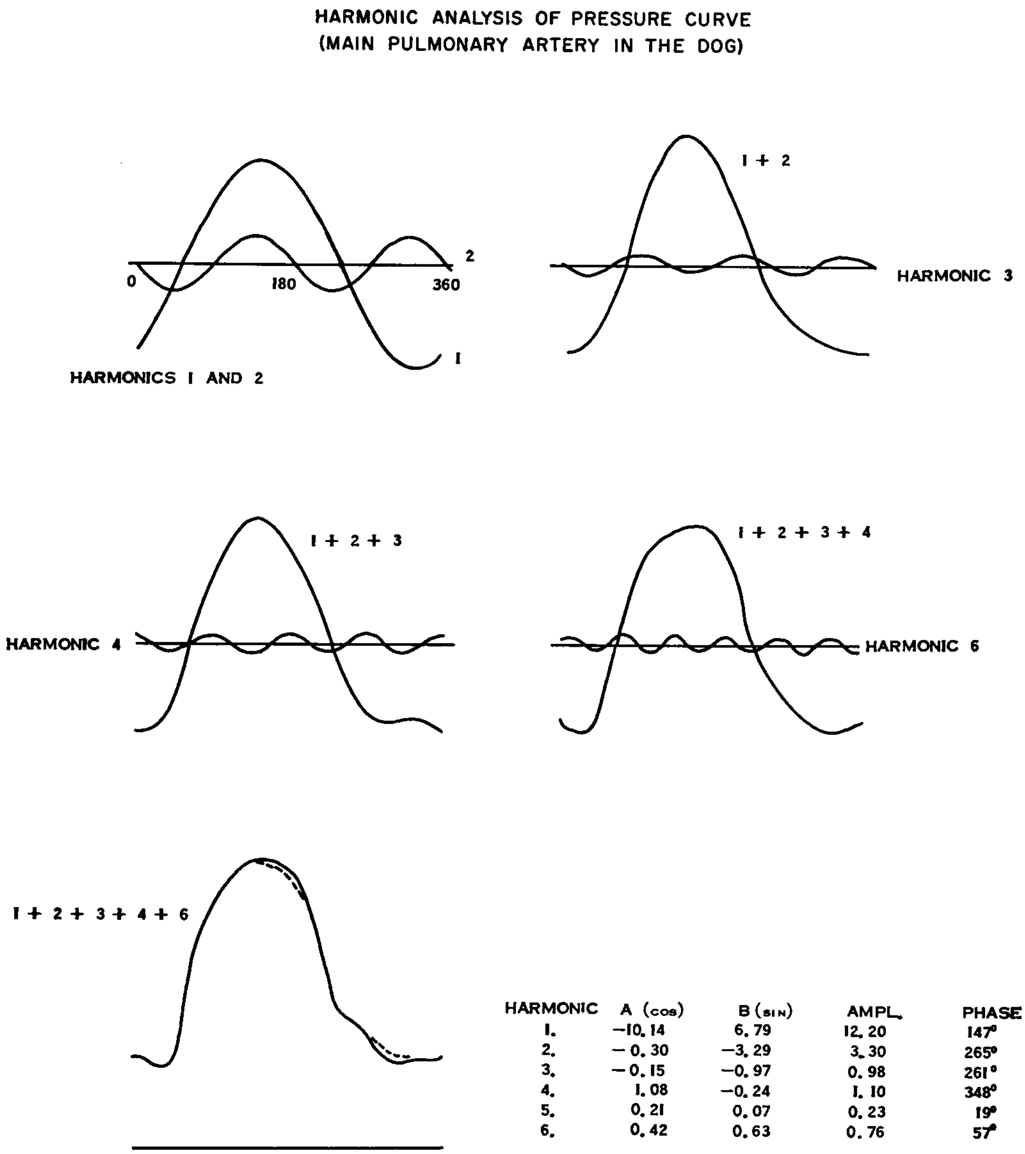
Representation of a Periodic Function. An arbitrary time function $f(t)$, which satisfies Dirichlet's conditions can be expanded in the trigonometric series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi n}{T} t \right) \tag{1}$$

where:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, dt \\ a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \frac{2\pi n t}{T}\, dt \\ b_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \frac{2\pi n t}{T}\, dt \end{aligned} \tag{2}$$

and T is the length of the interval (period) over which $f(t)$ is considered. The Dirichlet conditions are always satisfied for a biological system. Since equation (1) represents only a mathematical expression for a given function, any pressure or flow pulse may be decomposed in this manner into a number of sine or cosine waves (Fig. 1). Although the Fourier series in this figure describes the pulmonary pressure pulse over one cardiac cycle (the period over which it was analyzed) its usefulness in terms of system behavior would be severely restricted if the analysis of the next cardiac cycle would yield a different series. (This, of course, is a problem which is



HARMONIC	A (cos)	B (sin)	AMPL.	PHASE
1.	−10.14	6.79	12.20	147°
2.	−0.30	−3.29	3.30	265°
3.	−0.15	−0.97	0.98	261°
4.	1.08	−0.24	1.10	348°
5.	0.21	0.07	0.23	19°
6.	0.42	0.63	0.76	57°

FIGURE 1 Fourier analysis of a pulmonary artery pressure pulse. The figure illustrates the successively better approximation to the real curve, as more harmonic components are added. The experimental curve (dotted line) is closely duplicated by a resynthesis from the first 6 harmonics (lower left). The 5th harmonic was too small to be drawn on the figure.

not limited to Fourier series but is associated with any representation of an arbitrary time function and one of the basic reasons for the development of information theory and the analysis of random processes.) Hence, in order to prove the usefulness of Fourier series for the analysis of the circulatory (or respiratory) system one has to show that the length of the cardiac cycle is reasonably constant and that the shape and magnitude of subsequent pulses are identical. In other words, the system must be in a steady state, the effects of transient responses must have disappeared. Such a function is called periodic, and its periodicity is expressed by the relation

$$f(t + \alpha T) = f(t) \tag{3}$$

where T is the period and α is any positive or negative integer. The reciprocal of T is the fundamental frequency f:

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{4}$$

where ω is the fundamental angular frequency (radians/second). The coefficient a_0 in equation (1) represents the mean value of the function. The terms associated with the fundamental angular frequency ($a_1 \cos \omega t + b_1 \sin \omega t$) are commonly called the fundamental or first harmonic of the periodic function $f(t)$ and the remaining terms, whose frequencies are integer multiples of the fundamental, are referred to as the higher harmonics. The magnitude of the coefficients of the higher harmonics usually decreases with frequency so that a given function can be well approximated by only the first few terms of the series. This is true for the pressure and flow pulses in the cardiovascular system, since the magnitude of the higher harmonics falls within the noise level of the measuring equipment (Fig. 2) (2).

The Fourier series representation for a periodic function $f(t)$ can therefore be written as a finite number of terms:

$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos n\omega t + b_n \sin n\omega t) \tag{5}$$

By combining sine and cosine terms using well-known trigonometric identities equation (5) can be rewritten in terms of magnitude and phase angle:

$$f(t) = c_0 + \sum_{n=1}^{N} c_n \cos (n\omega t - \phi_n) \tag{6}$$

where:

$$c_n = (a_n^2 + b_n^2)^{1/2}$$
$$\phi_n = \tan^{-1} \frac{b_n}{a_n} \tag{7}$$

The coefficient c_n and the angle ϕ_n are, respectively, magnitude and phase angle of the n'th harmonic. c_0 represents, of course, the constant term, or the mean value of

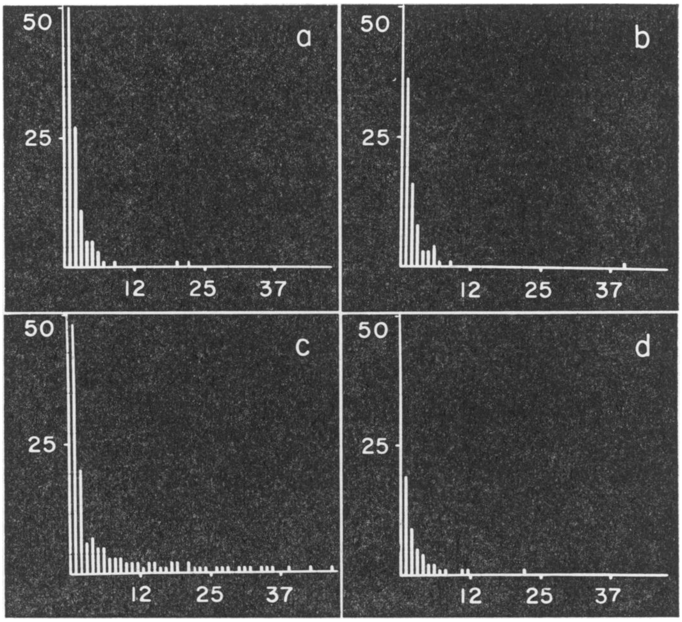


FIGURE 2 Fourier spectrum of flow and pressure in the mesenteric artery (*a, b*) and the respiratory system (*c, d*). The harmonic content of the vascular parameters is limited to the first 10 harmonics. That of air flow is somewhat broader (about 40 harmonics), although the magnitude of the higher harmonics is close to the inherent measurement error of the pneumotachograph.

the function. For many purposes it is more convenient to use the exponential form, because of the greater ease with which it can be manipulated. Using the Euler identities for sine and cosine, equation (6) can be written:

$$f(t) = \frac{1}{2} \sum_{n=-N}^{N} c_n e^{i(n\omega t - \phi_n)} \tag{8}$$

(For a derivation of the complex exponential form see, for example, reference 9.)

Periodicity. An erroneous period measurement of a periodic function leads to a violation of the definition, since under these conditions $f(t) \neq f(t + \alpha T)$. If subsequent periods were analyzed this error would become progressively worse. In addition to obtaining a spurious frequency spectrum one would also obtain a false mean value, which would change from cycle to cycle. The errors introduced by performing the Fourier analysis over an incorrect period are illustrated in Fig. 3. It will be seen that the difference between actual and analyzed period introduces additional harmonics, which are not present in the periodic function. In this particular example a pure sine wave was analyzed. When the analyzed period and the actual cycle length where identical (rel. period = 1) the analysis reproduced the pure sine wave exactly. The larger the difference between the two periods, the higher

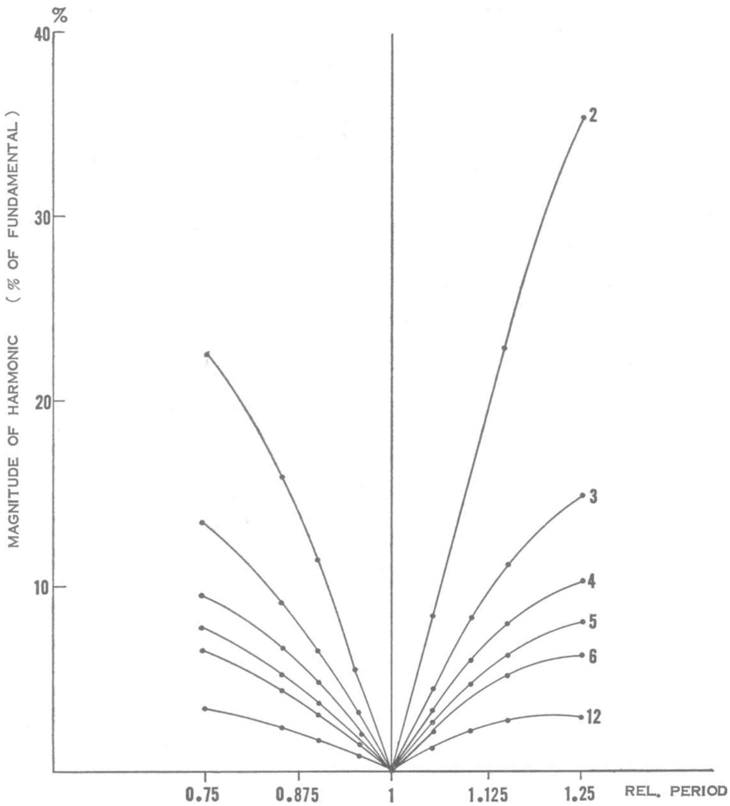


FIGURE 3 Spurious harmonic content introduced into a harmonic analysis if the periodic wave is analyzed over an interval which does not correspond to its actual period (rel. period $\neq$ 1.0).

becomes the spurious harmonic content of the sine wave. For example, if the actual period is 10% longer than the sampled period the analysis yields, in addition to the original sine wave, a second harmonic (amplitude 15% of the fundamental), a third harmonic (8%), a fourth harmonic (5%), and so on.

Linearity of the System. Although any curve can be described mathematically by a Fourier series, linearity of the system to be analyzed is a prerequisite for the use of the Fourier technique in a physical problem. For a linear system, the following principle is valid: if the forcing function can be separated into several parts, the response of the system is equal to the sum of the responses due to each part taken separately. This means that if the system is driven by a pure sine wave of frequency f, no pressure or flow components of another frequency are generated. In other words, there is no interaction between the various terms of the Fourier series. The relation between two variables can therefore be obtained quite simply. As an example, let the blood flow in an artery be represented by

$$\dot{Q}(t) = \frac{1}{2} \sum_{n=-N}^{N} \dot{Q}_n e^{i(n\omega t - \theta_n)} \tag{9}$$

and the pressure at the same point of the vessel by

$$P(t) = \frac{1}{2} \sum_{n=-N}^{N} P_n e^{i(n\omega t - \phi_n)} \tag{10}$$

then the impedance at that point for n'th harmonic can be written as

$$Z_n = \frac{P_n}{\dot{Q}_n} e^{i(\theta_n - \phi_n)} \tag{11}$$

Approximation by a Finite Number of Samples. In most practical applications the integrations from which the coefficients of the Fourier series are obtained (equation 2) are replaced by a finite summation process. The values of $f(t)$ are measured at K predetermined intervals of $\Delta t = T/K$ and equation (2) is replaced by

$$\begin{aligned} \bar{a}_n &= \frac{2}{K} \sum_{i=1}^{K} f_i \cos n \frac{2\pi}{K} i \\ \bar{b}_n &= \frac{2}{K} \sum_{i=1}^{K} f_i \sin n \frac{2\pi}{K} i \\ \bar{c}_n &= (\bar{a}_n + \bar{b}_n)^{1/2} \end{aligned} \tag{2a}$$

The question then arises as to how many samples are necessary to obtain a Fourier series of the desired degree of accuracy. This accuracy is determined by the number of terms (or harmonics) required for the series. The sampling theorem for a periodic function states (8) that if $f(t)$ is a periodic function and if all the Fourier coefficients vanish above the N'th harmonic, then $f(t)$ has only $2N + 1$ independent sampling values and these have a spacing of $T/(2N + 1)$ between them. This means that if the function to be analyzed has only 12 harmonics, 25 samples would be sufficient to obtain its Fourier series expansion. The coefficients obtained by finite summation ($\bar{a}_n$, $\bar{b}_n$) are equal to those obtained by integration (a_n, b_n) only if one takes two or more samples of the highest frequency component of the signal. The error introduced into the analysis if the number of samples is smaller than those specified by the sampling theorem can be estimated as follows:

Consider a periodic function $f(t)$ which contains no frequencies higher than the N'th harmonic. Suppose the function is sampled K times ($K < 2N + 1$) at equal intervals of time Δt. The value of the i'th sample is then, from equation (5):

$$f_i = a_0 + \sum_{j=1}^{N} \left\{ a_j \cos \left(j \frac{2\pi}{K} i \right) + b_j \sin \left(j \frac{2\pi}{K} i \right) \right\} \tag{12}$$

The highest harmonic that can be calculated from the K samples is

$$m = \frac{K - 1}{2} \tag{13}$$

We want to compare the coefficients $\bar{a}_n$ and $\bar{b}_n$ calculated from the K samples with the "true" coefficients a_n and b_n, obtained by integration (equation 2). Substituting equation (12) into equation (2a):

$$\bar{a}_n = \frac{2}{K} \sum_{i=1}^{K} \left[a_0 + \sum_{j=1}^{N} \left\{ a_j \cos\left(j \frac{2\pi}{K} i\right) + b_j \sin\left(j \frac{2\pi}{K} i\right) \right\} \right] \cos\left(n \frac{2\pi}{K} i\right) \quad (14)$$

Using the orthogonality relationships:

$$\sum_{i=1}^{K} \cos\left(n \frac{2\pi}{K} i\right) = 0 \qquad \text{for} \quad n = 1, 2, 3, \cdots$$

$$\sum_{i=1}^{K} \sin\left(j \frac{2\pi}{K} i\right) \cos\left(n \frac{2\pi}{K} i\right) = 0 \quad \text{for} \quad \begin{cases} j = 0, 1, 2, 3, \cdots \\ n = 0, 1, 2, 3, \cdots \end{cases} \quad (15)$$

equation (14) can be reduced to

$$\bar{a}_n = \frac{2}{K} \sum_{j=1}^{m} a_j \left\{ \sum_{i=1}^{K} \cos\left(j \frac{2\pi}{K} i\right) \cos\left(n \frac{2\pi}{K} i\right) \right\} + \frac{2}{K} \sum_{j=m+1}^{N} a_j \left\{ \sum_{i=1}^{K} \cos\left(j \frac{2\pi}{K} i\right) \cos\left(n \frac{2\pi}{K} i\right) \right\} \quad (16)$$

Since

$$\sum_{i=1}^{K} \cos\left(j \frac{2\pi}{K} i\right) \cos\left(n \frac{2\pi}{K} i\right) = 0 \quad \text{for} \quad j \neq n$$

$$= \frac{K}{2} \quad \text{for} \quad j = n \neq 0 \quad (17)$$

and

$$\cos\left[\{p(2m + 1) \pm n\} \frac{2\pi}{K} i \right] = \cos n \frac{2\pi}{K} i \quad \text{for} \quad p = 1, 2, 3, \cdots \quad \text{and} \quad K = 2m + 1$$

all terms in the series from $j = 1$ to m in equation (16) vanish except one in which $j = n$. Also all terms from $j = m + \text{v}$ to N vanish except those in which

$$j = (K \pm n), (2K \pm n), (3K \pm n)$$

Therefore

$$\bar{a}_n = a_n + a_{K-n} + a_{K+n} + a_{2K-n} + a_{2K+n} + \cdots \quad (18a)$$

and similarly

$$\bar{b}_n = b_n - b_{K-n} + b_{K+n} - b_{2K-n} + b_{2K+n} \cdots \quad (18b)$$

It will be seen from equations (18) and (13) that all the coefficients below the m'th are contaminated by higher harmonics present in the signal. This is called the aliasing phenomenon (6). The aliasing errors arise as a result of sampling the function at equal intervals of time which are larger than those specified by the sampling theorem. The frequency f_m, corresponding to the m'th harmonic, is called the folding frequency. While frequencies between $f = 0$ and $f = f_m$ are clearly distinct from one another, the frequencies higher than f_m are aliased with those below f_m. Each frequency, no matter how high, is indistinguishable from one within the frequency band from 0 to f_m. For example, if we want to analyze a pressure pulse at a heart rate of 60/min for 12 harmonics using 48 samples, we find that 60 cycle/sec noise within the measuring equipment introduces an aliasing error into the 12th harmonic $(60 = K + 12)$.

METHODS

Measurement of Variables. The experiments were carried out in anesthetized dogs (pentobarbital; 30 mg/Kg) and in excised dog aortas perfused with Krebs' solution. In the latter case all branches were carefully tied and pulsatile flow was provided by a servo-controlled pump, with variable sinusoidal output and frequency. The harmonic content of the output was continuously monitored. Intravascular and intrapleural pressures were measured with appropriate Statham strain gauges (series P 23 and SF-1). The natural frequency of the recording manometer system was between 90 and 200 cycle/sec for the P 23 (depending on the catheters) and about 2000 cycle/sec for the SF-1. Blood flow was monitored by means of an electromagnetic flowmeter (Medicon, model 2004), and air flow by a pneumotachograph. External vessel radius was estimated using induction coils, weighing 0.24 g per pair and excited with a frequency of 2400 cycle/sec from a Sanborn carrier amplifier. The frequency response of all the transducers and associated amplifying networks was flat (±5%) from zero to 20 cycle/sec, when subjected to a sinusoidal driving force yielding an analogue output from the whole system of 0.5 to 1 v. The static measurement errors were ±0.2 cm H_2O for the pressure transducers, ±0.1 cm^3/sec for the flowmeters, and $\pm 5 \times 10^{-4}$ cm for the estimate of radius changes. In order to obtain such a degree of accuracy extreme care is required in coupling the transducers to the system under investigation. All liquid-filled connections have to be made completely bubble-free, and the fit between vessel and flowmeter probe (or diameter coil) has to be snug and invariant. In our experience dynamic calibration of the measuring system before each experiment is essential if the data are subjected to a frequency analysis, although it may be very time consuming. In this context we would like to make a plea for more accurate and specific reporting of measurement errors. The statement: the measurement error is 5%, is meaningless if it is not related to a reference value. Thus, if a 5% error in the estimate mean pressure means a consistent error of, say, 5 cm H_2O, this would correspond to a 25% error in the estimate of the first or a 500% error in that of the sixth harmonic.

Analogue to Digital Conversion. The output of the transducers was fed to the A-D converter of a LINC digital computer (1). The converter has a maximal conversion speed of 25,000 samples/sec and accepts an analogue voltage from −1.0 to +1.0 v, with a resolution of 1:256; i.e., 7.8 mv. Zero suppression on the amplifiers was used to eliminate the dc components so that the oscillatory components could be amplified to the maximum compatible with the converter range. Under these conditions the resolution of the A-D converter is of the same order of magnitude as the noise level of the over-all system and the accuracy with which the calibrating instruments can be read (3).

Fourier Analysis. The Fourier analysis was carried out on line using equation (2*a*). The results presented in this paper are based on the analysis of more than 5000 cycles.

RESULTS AND DISCUSSION

Validity of Fourier Analysis.

Periodicity. The variation in heart rate and respiratory frequency observed in a number of anesthetized dogs is listed in Table I. The variation in rate observed over experimental periods ranging from 10 min to 4 hr is small and the standard deviation from the average frequency is of the order of 1 to 3%. According to Fig. 3, this would introduce a maximum spurious harmonic content of 6%

TABLE I

VARIATION OF HEART RATE AND RESPIRATORY FREQUENCY IN ANESTHETIZED DOGS

Dog	No. of cycles analyzed	Heart rate cycle/sec (mean ± SD)
1	33	3.02 ± 0.029
2	9	3.09 ± 0.049
3	20	2.78 ± 0.078
4	18	3.83 ± 0.027
5	29	3.65 ± 0.047
6	8	3.30 ± 0.100
7	17	2.49 ± 0.070
		Respiratory rate
		cycle/sec
a	6	0.632 ± 0.03
b	8	1.14 ± 0.03
c	8	0.265 ± 0.08

second, 3% third, 2% fourth, and fifth harmonic. In several experiments the Fourier analysis was carried out subsequently over 1, 2, 4, and 8 cycles. The results for two such runs obtained for flow and pressure in the superior mesenteric artery by analyzing 1, 2, 4, and 8 consecutive cycles on two runs, taken 1 min apart, are shown in part *b* of Table II. In this instance eight cardiac cycles covered about one respiratory cycle. The standard errors ($n = 8$) of both pressure and flow components (magnitude and phase) are smaller than the measurement error (see Methods section) and the steady-state assumption appears therefore to be valid. The sinus arrhythmia associated with respiratory activity appears to have little effect on the obtained results. There was no significant difference in both magnitude and phase of the harmonic coefficients, if they were obtained over one cardiac cycle chosen at random or over one respiratory cycle (comprising about 8 cardiac beats; see Table II and Fig. 4), although the respiratory pressure swing amounted to roughly 40% of the magnitude of the first harmonic (Fig. 4).

In unanesthetized animals irregular rhythms are more frequent and in this case

TABLE II

MODULUS c AND PHASE θ FOR THE HARMONICS OF MESENTERIC ARTERIAL PRESSURE FROM AN ANESTHETIZED DOG SELECTED AT RANDOM (PULSE FREQUENCY 2.5 CYCLE/SEC) SAMPLED AT 24, 96, 384, AND 768 SAMPLES PER CYCLE TABLE IIa AND OVER TWO, FOUR, AND EIGHT CARDIAC CYCLES TABLE II*b*. COMPARE WITH ONE CYCLE IN TABLE II*a*.

Table II*a*

No. samples	24		96		384		768			
Harmonic	c	$\theta°$	c	$\theta°$	c	$\theta°$	c	$\theta°$	c^*	$\theta°^*$
	cm H_2O		cm H_2O		cm H_2O		cm H_2O		cm H_2O	
1	13.2	170.5	13.3	170.5	13.4	171.1	12.9	171.5	13.22 ±0.09	170.8 ±0.25
2	5.8	319.0	5.7	318.5	5.7	320.0	5.6	320.0	5.68 ±0.04	319.5 ±0.4
3	3.3	70.6	3.3	72.0	3.3	73.8	3.0	73.1	3.25 ±0.03	72.4 ±0.6
4	1.2	170.5	1.3	177.2	1.3	174.4	1.1	173.8	1.25 ±0.03	172.6 ±0.8
5	1.3	258.5	1.3	258.1	1.3	261.6	1.2	261.0	1.32 ±0.07	260.3 ±0.9
6	1.5	39.6	1.5	39.6	1.5	42.8	1.4	44.4	1.46 ±0.03	40.9 ±1.3
7	0.7	169.0	0.6	166.0	0.6	166.7	0.6	172.0	0.61 ±0.03	169.1 ±3.1
8	0.3	301.7	0.2	248.0	0.2	247.5	0.2	270.0	0.2 ±0.04	245.3 ±12.8

Table II*b*

	c	$\theta°$	c	$\theta°$	c	$\theta°$	c^*	$\theta°^*$
No. cycles	2T		4T		8T			
Harmonic								
1	13.2	170.2	13.1	174.8	13.2	176.1	13.2 ± 0.14	172.9 ± 1.2
2	5.6	320.0	5.8	319.7	5.8	318.4	5.75 ± 0.06	319.4 ± 1.9
3	3.3	74.4	3.1	74.5	2.9	74.1	3.14 ± 0.08	73.7 ± 2.4
4	1.2	174.4	1.2	180.0	1.2	179.4	1.22 ± 0.06	176.3 ± 2.9
5	1.2	267.2	1.3	267.7	1.4	266.0	1.31 ± 0.04	264.5 ± 4.4
6	1.4	44.1	1.3	45.5	1.2	45.9	1.36 ± 0.05	44.1 ± 6.0

In the columns marked by asteriks the over-all means and standard errors for the following data are listed: (*a*) two cardiac cycles were analyzed at four sampling rates each; (*b*) two sets of data where analyzed over two, four, and eight periods each, using a sampling rate of 96 samples per period.

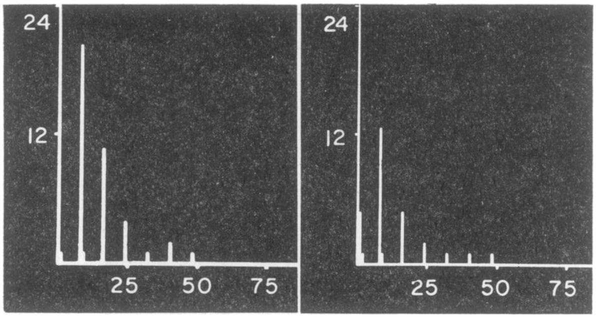


FIGURE 4 Fourier spectrum of pressure and flow in the mesenteric artery, obtained by the analysis of 8 consecutive cycles. (The 8th harmonic represents the fundamental of the cardiac cycle.) The 1st and 2nd harmonics are due to respiratory activity, and are considerably larger in the pressure pulse than in the flow pulse (pulse frequency 2.5 cycle/sec).

it might be advisable to sample several cardiac cycles, preferably over one respiratory cycle, in order to obtain a representative sample for the analysis of the system. This approach has the additional advantage that it permits the evaluation of respiratory effects.

Linearity. We are concerned here only with linearity of the vascular system over the range of magnitude of the pulsatile components at a given mean pressure. Over this range, the pulsatile components of strain are small, and as Noll (10) pointed out, it would be permissible to assume linearity on the basis of smoothness alone. We attempted to detect nonlinear effects by perfusing excised dog aortas forward and backward, using a pump with variable sinusoidal stroke volume and variable frequency, as described under the Methods section. Since the radius of the abdominal aorta is about half that of the ascending aorta, we expected that nonlinearities would become more apparent, since for forward flow we were perfusing a converging and for backward flow a diverging tube. The harmonic content of the pump output was monitored and pressures were measured at at least four points along the vessel. In all experiments only the harmonic generated by the pump were observed in the system. Driving the system with different stroke volumes, covering an oscillatory pressure range of ± 100 cm H_2O about the mean distending pressure of 150 cm H_2O, showed no differences in the calculated impedances within the accuracy of our measurements.

It seems, therefore, that at least with our present accuracy of measurement, nonlinear effects are not detectable and that the arterial system can safely be treated as linear as far as physiological oscillatory components in the arterial system are concerned. Similar conclusions based on Fourier analysis of pressure and flow at varying heart rates have been reached for the pulmonary vasculature of the dog (5), and based on a mathematical treatment of experimental dye dilution curves by

Bassingthwaighte (4). In the cases where several subsequent cardiac cycles were analyzed, no subharmonics were found, except those associated with respiratory activity (Fig. 4).

Accuracy of Fourier Analysis.

Errors due to measurement of period. It has already been pointed out (Fig. 3) that erroneous measurement of cycle length may lead to considerable errors in a harmonic analysis. If the heart rate is fairly regular this should not represent a serious problem, since even on chart recorders the cycle length can be determined to an accuracy of 3%. In the LINC the accuracy of the period determination is better than 8 parts in 10,000.

Errors due to the sampling procedure. Since the calculation of the Fourier coefficient is a rather lengthy procedure (even for a computer), it is of interest to determine the minimum number of samples required for an adequate representation of the pressure and flow pulse. This minimum number depends both on sampling and aliasing errors. As pointed out in the section on theory, the theoretical minimum sampling rate is determined by the highest frequencies present. In practice it is sometimes advisable to sample at a rate five to ten times higher than that predicted by the sampling theorem if the sampling device introduces significant errors (6). Even then the presence of noise in the system may introduce aliasing errors (see the 40th harmonic in Fig. 2*b*, representing 60 cycle/sec noise).

Table II *a* lists the mean and standard errors obtained for flow and pressure in the mesenteric artery during two different runs, sampled subsequently at 96, 192, 384, and 768 samples per cycle. As theory predicts, the increase in sampling rate does not increase the accuracy of the determination of the Fourier coefficient. The fact that the amplitude of all the harmonics is independent of the sampling rate over the range investigated indicates that both aliasing errors (due to noise) and sampling errors are negligible.

For the higher harmonics ($n > 6$) the error in magnitude and phase is rather larger and the calculation of Z_n (equation 11) becomes, therefore, unreliable at these frequencies. The error in the calculated angles will be largest, if the *a* and *b* coefficients (equation 7) are both small and of comparable value [$\tan \phi \sim 1$ or $\phi = (45 \pm n\pi/2)$]. This accuracy can be improved somewhat by improving the resolution of the A-D converter but *not* by increasing the number of samples. The major part of any error in either magnitude or phase of a harmonic is still due to the inadequacy of the presently available transducers.

SUMMARY

Using careful measurement techniques the value of Fourier analysis of the cardiovascular and respiratory systems has been evaluated.

It was found that the two basic postulates for Fourier analysis, namely periodicity

and linearity, are usually satisfied for the oscillatory components. Errors introduced by deviation from these two conditions are within the range of measurement errors in normal, anesthetized dogs. It is suggested that in cases of marked arrhythmia a larger number of cycles be sampled.

The magnitude of errors due to faulty determination of cycle length, sampling techniques, and aliasing has been estimated and found to be negligible, provided proper precautions are taken. No increase in accuracy is to be expected if the sampling rate is increased above that required by the sampling theorem.

The results indicate that Fourier analysis is a valid technique for the investigation of the oscillatory components of the circulatory and respiratory systems. Its accuracy depends primarily upon the reliability of the transducer amplifying sampling system. Dynamic calibration of the measuring system before each experiment is a prerequisite for successful analysis. Sampling accuracy must be compatible with measurement accuracy. Sophisticated computer programming cannot compensate for sloppy experimental techniques.

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